Claim: Any solvable ODE can be turned into an exact ODE by multiplying by some integrating factor \( \mu(x,y) \).

Recall:

1. \( M(x,y) \, dx + N(x,y) \, dy = 0 \) is an exact ODE if \( \exists \phi(x,y) \) s.t.
   \[
   \begin{align*}
   f_x(x,y) &= M(x,y) \\
   f_y(x,y) &= N(x,y)
   \end{align*}
   \]

2. If \( M_y = N_x \), then \( M(x,y) \, dx + N(x,y) \, dy = 0 \) is exact.
   (If \( M \) and \( N \) are continuously differentiable on some open set, the converse also holds.)

We solved

\[
y' = \frac{-2x e^y - 5}{x^2 e^y + \cos y}
\]

But this is equivalent to

\[
y' = \frac{-2x - 5e^y}{x^2 + e^{-y} \cos y}
\]

Can we solve this as an exact equation?

\[
(2x + 5e^{-y}) \, dx + (x^2 + e^{-y} \cos y) \, dy = 0
\]

We need to put this back.

But, if we simply were presented with

\[
(2x + 5e^{-y}) \, dx + (x^2 + e^{-y} \cos y) \, dy = 0
\]

How would we figure this out?

Integrating factor: \( \mu(x,y) \)

\[
\begin{align*}
\frac{\partial}{\partial y} M + \mu M_y &= \mu_x N + \mu N_x \\
M_y M + \mu N_y &= \mu_x N + \mu N_x
\end{align*}
\]

Want to find \( \mu \) s.t.

These are equal.
In general this is difficult, but if \( \mu = g(y) \) (just a function of \( y \), not of \( x \)) then \( \mu_x = 0 \), so

\[
\mu_y M + \mu M_y = 0 + \mu N_x
\]

\[
\Rightarrow \quad \mu_y M = \mu (N_x - M_y)
\]

\[
\Rightarrow \quad \frac{\mu_y}{\mu} = \frac{N_x - M_y}{M}
\]

This is only a function of \( y \), so

\[
\Rightarrow \quad \int \frac{\mu}{\mu} \, dy = \int \frac{N_x - M_y}{M} \, dy
\]

\[
\Rightarrow \quad \mu(y) = e^{\frac{\int (N_x - M_y) \, dy}{M}}
\]

\[\boxed{(2x + 5e^{-y}) \, dx + (x^2 + e^{-y} \cos y) \, dy = 0}\]

\[
M_y(x,y) = -5e^{-y} \quad N_x(x,y) = 2x
\]

\[
\Rightarrow \quad \frac{N_x - M_y}{M} = \frac{2x + 5e^{-y}}{2x + 5e^{-y}} = 1
\]

(trivially a function of \( y \).)

\[\Rightarrow \quad \mu(y) = e^{\int \frac{N_x - M_y}{M} \, dy} = e^{y}
\]

\[\Rightarrow \quad e^{y}(2x + 5e^{-y}) \, dx + e^{y}(x^2 + e^{-y} \cos y) \, dy = 0 \quad \text{(it's actually the ODE we solved last time)}\]

We can solve it like before!

\[
f(x,y) = \int e^{y}(2x + 5e^{-y}) \, dx + h(y) = x^2 e^{y} + 5x + h(y)
\]

\[
\Rightarrow \quad f(x,y) = e^{y}(x^2 + 5x + 5y) + \mu(x) = x^2 e^{y} + 5x + 5y + g(x)
\]

We assumed \( \mu \) was only a function of \( y \). It might not be, in which case the above won't work. We could have also assumed \( \mu \) is only a function of \( x \). This leads to

\[
\mu(x) = e^{\int \frac{N_x - M_y}{N} \, dx}
\]

(see the homework question).

It is also possible that \( \mu \) is not just a function of \( x \) or \( y \) separately, in which case finding it can be more difficult.